Journal of Statistical Physics, Vol. 102, Nos. 1/2, 2001

Hydrodynamic Profiles for the Totally Asymmetric Exclusion Process with a Slow Bond

Timo Seppäläinen[1]

Received April 10, 2000; final July 21, 2000

We study a totally asymmetric simple exclusion process where jumps happen at rate one, except at the origin where the rate is lower. We prove a hydrodynamic scaling limit to a macroscopic profile described by a variational formula. The limit is valid for all values of the slow rate. The only assumption required is that a law of large numbers holds for the initial particle distribution. This allows also deterministic initial configurations. The hydrodynamic description contains as an unknown parameter the macroscopic rate at the origin, which is strictly larger than the microscopic slow rate. The limit is proved by the variational coupling method.

KEY WORDS: Exclusion with blockage; slow bond; hydrodynamic limit.

1. INTRODUCTION

The exclusion process consists of particles executing independent random walks on a graph, subject to the exclusion interaction that prevents two particles from occupying the same vertex. In this paper we look at the version where the underlying graph is the integer lattice $\mathbf{Z}$ with nearest-neighbor bonds. The particles take only nearest-neighbor steps to the right, and a jump is permitted only if the next site to the right is vacant. In the spatially homogeneous case jumps across all nearest-neighbor bonds happen at exponential rate one. This model goes by the name TASEP, or totally asymmetric simple exclusion process.

[1] Department of Mathematics, Iowa State University, Ames, Iowa 50011; e-mail: seppalai@iastate.edu.

0022-4715/01/0100-0069$19.50/0

In the early 1990's Janowsky and Lebowitz[1,2] introduced an interesting variant of TASEP. Jumps across a specific bond, say $\{0, 1\}$, happen at a rate r strictly lower than the rate 1 everywhere else. Imagine for example a tollbooth on a single lane highway that slows down the flow of cars. For the slow bond model one investigates the same questions as for homogeneous TASEP, such as large scale behavior and invariant distributions. Simulations and partial results are available, but crucial results remain open.

In the present paper we prove a hydrodynamic limit for the slow bond model. In one sense our result is complete and general. It is valid for all values $r \in (0, 1)$ of the slow rate. We need no extra assumptions on the initial distribution of the particles, only that the law of large numbers for the density profile be valid at time zero. But the hydrodynamic limit cannot make the most interesting distinction, namely whether the slow bond disturbs hydrodynamic profiles for all values $r < 1$. This question can be resolved only through sharper control of particle-level evolution.

The macroscopic effect of the slow bond can be described like this: Corresponding to the microscopic rate profile (rate identically 1 at all sites except 0, and rate equal to r at 0) there is a macroscopic rate profile $\lambda(x)$ defined for $x \in \mathbf{R}$ such that $\lambda(x) = 1$ for $x \neq 0$, and $\lambda(0) \in (r, 1]$. The macroscopic evolution of the particle density is the solution of an optimal control problem whose running cost depends on the space variable x through the function $\lambda(x)$. If $\lambda(0) = 1$, there is no visible macroscopic disturbance from the slow bond. The interesting open question becomes: Is $\lambda(0) < 1$ for all $r < 1$?

We give a precise definition of the quantity $\lambda(0)$ in terms of a last-passage growth model whose association with TASEP is well-known. To prove the hydrodynamic limit we use the variational coupling method initiated in ref. 3 which, when it applies, gives laws of large numbers without any knowledge of invariant distributions. The proof extends to the situation with multiple slow bonds, and also works for more general exclusion processes that admit K particles per site.

In addition to the Janowsky–Lebowitz papers mentioned above, there is a handful of other related work. Covert and Rezakhanlou[4] derived a bound for the critical value of the slow rate by approximating the slow bond model with an exclusion process whose rates vary more regularly in space. Liggett's[5] recent monograph discusses the slow bond model and proves bounds for the critical value of the rate. As for Janowsky and Lebowitz, the approach is through finite systems with open boundaries, with system size tending to infinity. The case of a zero-range process (ZRP) with a slow site was treated by Landim.[6] In ZRP particles accumulate at the slow site and produce a point mass in the hydrodynamic profile. The

ZRP is a more tractable model because it retains product-form invariant distributions even when rates lose spatial homogeneity.

Our paper is organized as follows. The main result is the hydrodynamic limit Theorem 2.2. On the way to it we define the macroscopic rate $\lambda(0)$ in terms of the growth model, and state some bounds for it in Theorem 2.1. As corollaries to Theorem 2.2 we compute the macroscopic profiles that evolve from constant initial profiles, characterize the macroscopically invariant profiles in the range $[\rho^*, 1-\rho^*]$ between the lower and upper critical densities, and prove one property of invariant measures. These results are in Section 2. The remainder of the paper contains the proofs.

2. RESULTS

The process operates according to these rules: Indistinguishable particles occupy the sites of the one-dimensional integer lattice $\mathbf{Z}$. Each site has at most one particle, so the state of the process is described by the occupation numbers $\eta = (\eta_i)_{i \in \mathbf{Z}}$, where $\eta_i = 1$ if site i is occupied, and 0 if site i is empty. Particles take nearest-neighbor steps to the right, subject to the exclusion rule, at exponential rate 1, with one exception: jumps from site 0 to site 1 happen at rate r. The rate r is a fixed constant in the range $(0, 1]$. On the compact state space $\{0, 1\}^{\mathbf{Z}}$, the dynamics has the infinitesimal generator

$$Lf(\eta) = \sum_{i \neq 0} \eta_i(1-\eta_{i+1})[f(\eta^{i,i+1}) - f(\eta)] + r\eta_0(1-\eta_1)[f(\eta^{0,1}) - f(\eta)] \tag{2.1}$$

where $\eta^{i,i+1} = \eta - \delta_i + \delta_{i+1}$ denotes the configuration that results after a single particle jumps from i to $i+1$. When $r = 1$ this is the generator of TASEP, the totally asymmetric simple exclusion process.

We prove a hydrodynamic limit for this process in the usual Euler scale, for all values of r. As is well-known, when $r = 1$ the macroscopic particle density $\rho(x, t)$ obeys the scalar conservation law

$$\rho_t + f_0(\rho)_x = 0 \tag{2.2}$$

with current

$$f_0(\rho) = \rho(1-\rho) \tag{2.3}$$

Starting with Rost[7] in 1981, this hydrodynamic limit of the space-homogeneous TASEP has gone through many stages of generalization and refinement. See ref. 8, Chapter 8; ref. 5, Part III, and their notes and references.

The slow bond restricts the range of admissible currents. In TASEP the range of currents is $[0, 1/4]$, and the maximal current $1/4 = f_0(1/2)$ occurs at density $\rho = 1/2$. In the slow bond model the maximal current is some value $J_{\max}(r) \leqslant 1/4$. We know rigorously that $J_{\max}(r) < 1/4$ if r is small enough. When this happens, the densities ρ around 1/2 for which $f_0(\rho) > J_{\max}(r)$ become inadmissible. This we can prove on the hydrodynamic scale.

Let $\kappa(r) = J_{\max}(r)^{-1}$ be the reciprocal of the maximal rate. Thus $\kappa(r)$ is a nonincreasing function of $r \in (0, 1]$, with values in the range $[4, \infty)$ and $\kappa(1) = 4$. Since we do not know the exact dependence of $\kappa(r)$ on r, we shall give it a precise definition through a well-known growth model.

2.1. Definition of $\kappa(r)$

Consider the following last-passage growth model on the first quadrant of the plane. For $(i, j) \in \mathbf{N}^2$, let $Y_{i,j}$ be i.i.d. exponentially distributed random variables with common expectation $E[Y_{i,j}] = 1$. Let the deterministic weights $w_{i,j}$ be given by

$$w_{i,j} = \begin{cases} 1, & i \neq j \\ 1/r, & i = j \end{cases} \tag{2.4}$$

For $n = 1, 2, 3,...$, let the passage time of site (n, n) be

$$T_{n,n} = \max_{\pi} \sum_{k=1}^{2n-1} w_{i_k, j_k} Y_{i_k, j_k} \tag{2.5}$$

where the maximum is over paths $\pi = \{(1, 1) = (i_1, j_1), (i_2, j_2),..., (i_{2n-1}, j_{2n-1}) = (n, n)\}$ that take steps only to the right and up: for each k,

$$(i_k, j_k) - (i_{k-1}, j_{k-1}) = (1, 0) \qquad \text{or} \qquad (0, 1) \tag{2.6}$$

An obvious superadditivity holds for the random variables $\{T_{n,n}\}$. One can derive moment bounds (see for example Theorem 6.3 in ref. 9, or Proposition 5.1 in ref. 10) sufficient for Kingman's[11] subadditive ergodic theorem to obtain a strong law of large numbers: There exists a constant $\kappa(r)$ such that

$$\lim_{n \to \infty} \frac{1}{n} T_{n,n} = \kappa(r) \qquad \text{a.s.} \tag{2.7}$$

This limit is taken as the definition of $\kappa(r)$. It is known that $\kappa(1)=4$. This is a special case of the full interface result: for $(x, y) \in \mathbf{R}_+^2$

$$\lim_{n\to\infty} \frac{1}{n} T_{[nx],\,[ny]} = (\sqrt{x} + \sqrt{y}\,)^2 \qquad \text{a.s. in the case } r=1 \tag{2.8}$$

Proofs of (2.8) can be found in refs. 7 and 3.

Currently we have this information on $\kappa(r)$:

Theorem 2.1. The function $\kappa(r)$, $0<r\leqslant 1$, is continuous and non-increasing. It satisfies these bounds:

$$0 \leqslant \kappa(r_1) - \kappa(r_2) \leqslant \frac{1}{r_1} - \frac{1}{r_2} \qquad \text{for} \quad r_1 < r_2$$

and

$$\max\left\{4, \frac{3}{2} + \frac{r^2 + 2(1+r)}{2r(1+r)}\right\} \leqslant \kappa(r) \leqslant 3 + \frac{1}{r} \tag{2.9}$$

The proof shows that neither bound in (2.9) is optimal. Janowsky and Lebowitz[2] give the bound $J_{\max}(r) \leqslant r(1-r)$ for $r \leqslant 1/2$. (For the proof, see p. 277 in ref. 5.) For $\kappa(r)$ this gives

$$\kappa(r) \geqslant \frac{1}{r(1-r)} \qquad \text{for} \quad r \leqslant 1/2 \tag{2.10}$$

This improves (2.9) in the range $r \in [(\sqrt{17}-1)/8, 1/2]$. However, the definition of $J_{\max}(r)$ in refs. 2 and 5 is not the same as ours, which is tied to the hydrodynamic limit. In refs. 2 and 5, $J_{\max}(r)$ is the limiting stationary current of a finite system with open boundaries, as the size of the system tends to infinity. Of course the two quantities ought to be the same. Once optimal bounds are found with some approach, it will be of interest to verify that the different definitions are the same.

Let us define

$$r^* = \inf\{r \in (0, 1] : \kappa(r) = 4\}$$

It is an open problem whether $r^*=1$. In other words, does the macroscopic passage time $\kappa(r)$ rise strictly above $\kappa(1)=4$ as soon as $r<1$? Equivalently, is there a forbidden range of densities around 1/2 as soon as $r<1$? Simulations by Janowsky and Lebowitz[2] suggest that such is the case. The lower bound in (2.9) gives $r^* \geqslant (\sqrt{41}-3)/8 \approx 0.425$, while (2.10) gives $r^* \geqslant 1/2$.

2.2. The Hydrodynamic Limit

Let ρ_0 be a given measurable function on $\mathbf{R}$ such that $0 \leqslant \rho_0(x) \leqslant 1$ for all $x \in \mathbf{R}$. This is the initial macroscopic profile. Let v_0 be the antiderivative of ρ_0 defined by

$$v_0(0) = 0, \qquad v_0(b) - v_0(a) = \int_a^b \rho_0(x)\, dx \tag{2.11}$$

Define the "macroscopic rate profile" $\lambda(x)$ as

$$\lambda(x) = \begin{cases} 1, & x \neq 0 \\ \dfrac{4}{\kappa(r)}, & x = 0 \end{cases} \tag{2.12}$$

where $\kappa(r)$ is defined by (2.7). For $x \in \mathbf{R}$, let

$$g_0(x) = \sup_{0 \leqslant \rho \leqslant 1} \{ f_0(\rho) - x\rho \}$$

denote the Legendre conjugate of f_0. It is given by

$$g_0(x) = \begin{cases} -x, & x \leqslant -1 \\ (1/4)(1-x)^2, & -1 \leqslant x \leqslant 1 \\ 0, & x \geqslant 1 \end{cases} \tag{2.13}$$

For $x \in \mathbf{R}$ and $t \geqslant 0$, define $v(x, 0) = v_0(x)$, and for $t > 0$,

$$v(x, t) = \sup_{w(\cdot)} \left\{ v_0(w(0)) - \int_0^t \lambda(w(s))\, g_0\left(\frac{w'(s)}{\lambda(w(s))} \right) ds \right\} \tag{2.14}$$

The supremum is over piecewise C^1 paths $w: [0, t] \to \mathbf{R}$ that satisfy $w(t) = x$. In Section 5.1 we give a formula for the path $w(\cdot)$ that minimizes the integral part inside the braces for a given initial point $w(0) = q$.

For each fixed t, $v(\cdot, t)$ is a Lipschitz function. Its x-derivative

$$\rho(x, t) = \frac{\partial}{\partial x} v(x, t) \tag{2.15}$$

represents the macroscopic density profile of the particles.

If $\lambda(x) = 1$ for all x, (2.14)–(2.15) give the entropy solution of (2.2) with initial data ρ_0. This is the density profile of space-homogeneous TASEP.

Formula (2.14) expresses the sense in which $\lambda(x)$ can be regarded as the macroscopic rate profile. From (2.9) we get

$$\lambda(0) \geqslant \frac{4r}{3r+1}$$

This implies that $\lambda(0) > r$ for $r < 1$. In other words, the microscopic averaging does *not* simply reproduce the rate r at the macroscopic level.

For the hydrodynamic limit, assume that we have constructed a sequence of exclusion processes $\eta^n(t) = (\eta_i^n(t) : i \in \mathbf{Z})$ for times $t \geqslant 0$, where $n = 1, 2, 3...$ is the index of the sequence. The initial distributions of the processes are arbitrary, subject to the condition that this weak law of large numbers is valid:

$$\text{for all} \quad a < b \quad \text{and} \quad \varepsilon > 0$$
$$\lim_{n\to\infty} P^n \left\{ \left| \frac{1}{n} \sum_{i=[na]+1}^{[nb]} \eta_i^n(0) - \int_a^b \rho_0(x)\, dx \right| \geqslant \varepsilon \right\} = 0 \tag{2.16}$$

We wrote P^n for the probability measure on the probability space of the process η^n. Let $J_i^n(t)$ denote the number of particles that have made the jump from site i to $i+1$ in the time interval $[0, t]$, in the process $\eta^n(\cdot)$.

Theorem 2.2. Let the slow rate r be any number in $(0, 1]$. Under assumption (2.16), these weak laws of large numbers hold at macroscopic times $t > 0$: for all real numbers $a < b$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P^n\{ |n^{-1} J_{[na]}^n(nt) - (v_0(a) - v(a, t))| \geqslant \varepsilon \} = 0 \tag{2.17}$$

and

$$\lim_{n\to\infty} P^n \left\{ \left| \frac{1}{n} \sum_{i=[na]+1}^{[nb]} \eta_i^n(nt) - \int_a^b \rho(x, t)\, dx \right| \geqslant \varepsilon \right\} = 0$$

where $v(x, t)$ is defined by (2.14) and $\rho(x, t) = v_x(x, t)$.

Let us derive some corollaries of Theorem 2.2. When $\lambda(0) < 1$ there is a critical density

$$\rho^* = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \lambda(0)} \tag{2.18}$$

determined by the current condition $f_0(\rho^*) = \lambda(0)/4$. The slow bond disturbs the hydrodynamic profile only for densities in the range $\rho \in (\rho^*, 1 - \rho^*)$.

When $\lambda(0)=1$ [or, equivalently, the passage time $\kappa(r)=4$], $\rho^*=1/2$ and the interval $(\rho^*, 1-\rho^*)$ is empty.

As a first application of Theorem 2.2 we derive the macroscopic evolution of constant initial profiles.

Corollary 2.1. Assume the initial distribution of the particles is chosen so that (2.16) is satisfied for the constant profile $\rho_0(x)\equiv\rho$. Then the macroscopic limits in Theorem 2.2 are described as follows:

Case 1: $\rho\in[0,\rho^*]\cup[1-\rho^*,1]$. Then $v(x,t)=\rho x-t\rho(1-\rho)$ and macroscopically we see a constant density $\rho(x,t)\equiv\rho$, exactly as for the homogeneous TASEP.

Case 2: $\lambda(0)<1$ and $\rho\in(\rho^*,1-\rho^*)$. The behavior changes around the blockage at the origin:

$$v(x,t)=\begin{cases}(1-\rho^*)\,x-t\lambda(0)/4, & -t(\rho-\rho^*)\leqslant x\leqslant 0\\ \rho^*x-t\lambda(0)/4, & 0<x\leqslant t(1-\rho^*-\rho)\end{cases}\tag{2.19}$$

Case 1 is still valid for x outside the range in (2.19). Correspondingly, the density profile has constant segments of upper and lower critical densities around the origin:

$$\rho(x,t)=\begin{cases}\rho, & x<-t(\rho-\rho^*)\quad\text{or}\quad x>t(1-\rho^*-\rho)\\ 1-\rho^*, & -t(\rho-\rho^*)<x<0\\ \rho^*, & 0<x<t(1-\rho^*-\rho)\end{cases}\tag{2.20}$$

The reason for this behavior is that the maximal current permitted by the blocked system is $\lambda(0)/4=\rho^*(1-\rho^*)$, while the unblocked TASEP has current $f_0(\rho)=\rho(1-\rho)$. The blockage does not disturb the system unless the system tries to transport particles at a current above $\lambda(0)/4$. This happens if $\rho(1-\rho)>\lambda(0)/4$, which is equivalent to $\rho\in(\rho^*,1-\rho^*)$.

Corollary 2.2. Suppose $\lambda(0)<1$ so that $\rho^*\in(0,1/2)$. Let ρ_0 be a macroscopic profile that is piecewise continuous in each bounded interval, and satisfies $\rho^*\leqslant\rho_0(x)\leqslant 1-\rho^*$. Suppose ρ_0 is invariant under the macroscopic dynamics, so that $\rho(x,t)=\rho_0(x)$ for a.e. x, for all times t. Then Lebesgue-almost everywhere $\rho_0(x)\in\{\rho^*,1-\rho^*\}$, and ρ_0 takes 0–3 jumps according to these restrictions: a jump from $1-\rho^*$ to ρ^* can occur only at the origin, while jumps from ρ^* to $1-\rho^*$ can occur at any $x\in\mathbf{R}$.

We can control ρ_0 only almost everywhere because a macroscopic density profile $\rho(\cdot,t)$ is determined only through its integral $v(\cdot,t)$. The upward jumps from ρ^* to $1-\rho^*$ are the usual entropy shocks of Eq. (2.2). The slow bond permits a downward jump from $1-\rho^*$ to ρ^* at the origin, which violates the entropy condition for the concave current f_0.

By Corollary 2.2 there cannot be a macroscopically invariant profile strictly in the range $(\rho^*, 1-\rho^*)$. Consequently, the process cannot have an invariant probability distribution for which a macroscopic profile exists in the sense of (2.16) and lies in the range $(\rho^*, 1-\rho^*)$. The interesting open problem is to prove the existence of an invariant probability measure that corresponds to the macroscopically invariant non-entropy shock profile $\rho_0(x) = (1-\rho^*)\,\mathbf{1}\{x<0\} + \rho^*\mathbf{1}\{x>0\}$.

About invariant measures we have this to say:

Corollary 2.3. For any value r of the slow rate, and for any $\rho \in [0, \rho^*] \cup [1-\rho^*, 1]$, there exists an invariant distribution μ for the process such that

$$\mu\{\eta_i = 1, \eta_{i+1} = 0\} = \begin{cases} r^{-1}\rho(1-\rho), & i=0 \\ \rho(1-\rho), & i \neq 0 \end{cases} \tag{2.21}$$

In particular, suppose the initial distribution of the process is the Bernoulli distribution α_0 with marginals

$$\alpha_0\{\eta_i = 1\} = 1-\rho^* \quad \text{for } i \leqslant 0, \qquad \text{and} \qquad \alpha_0\{\eta_i = 1\} = \rho^* \quad \text{for } i > 0$$

Let α_t denote the distribution at time $t > 0$. Then any limit point μ of the time averages $t^{-1}\int_0^t \alpha_s\, ds$ satisfies (2.21) with $\rho = \rho^*$.

2.3. Remarks and Extensions

K-Exclusion and Multiple Slow Bonds. Our proof uses the method of ref. 10. As in that paper, we can prove Theorem 2.2 also for generalized exclusion processes that allow K particles per site, instead of just one particle. The result would be qualitatively the same, but with a different growth model in Section 2.1 and correspondingly different $\lambda(0)$. With suitable large deviation estimates, as in ref. 10, we envision that the weak law of large numbers in Theorem 2.2 could be strengthened to a strong law of large numbers (convergence for almost every realization of the microscopic evolution).

A natural extension is to consider finitely many macroscopically separated slow bonds. Suppose that for the nth process η^n, jumps from site $[nx_k]$ occur at rate r_k for some set $x_1 < \cdots < x_m$ of macroscopic space points. Theorem 2.2 remains unchanged. The change appears in the definition (2.12) of the macroscopic rate profile: the function $\lambda(\cdot)$ would take on the values $\lambda(x_k) = 4/\kappa(r_k)$, and $\lambda(x) = 1$ elsewhere.

Hamilton–Jacobi Equations. (2.14) describes $v(x, t)$ as the value function of an optimal control problem with running cost $\lambda(w(s))\, g_0(w'(s)/\lambda(w(s)))$. If the function λ had sufficient regularity, standard theory would imply that $v(x, t)$ is the unique viscosity solution of the Cauchy problem

$$v_t + \lambda(x)\, f_0(v_x) = 0, \qquad v(\cdot, 0) = v_0 \tag{2.22}$$

(See Chapt. 10 in ref. 12.) One can directly check that the solutions in Cases 1 and 2 of Corollary 2.1 satisfy (2.22) except at corner points. But there is currently no theory about existence and uniqueness of solutions of Hamilton–Jacobi equations with a discontinuity of the type that $\lambda(x)$ has at 0. The presently available approaches to discontinuous equations involve smoothing out the discontinuity with a mollifier, see ref. 13 and its references. But once λ has been convolved with a mollifier, the discontinuity at 0 is lost. So a different approach is required for our problem.

The Growth Model. The connection between TASEP and the homogeneous version of the growth model of Section 2.1 goes back to Rost's 1981 paper.[7] The last-passage formulation was first utilized for hydrodynamic limits and large deviations in refs. 3 and 14. The connection between exclusion and the growth model will appear in Section 4.

An additional motivation for the definition of $\kappa(r)$ in terms of the path model is this: analysis of models of this type with tools from combinatorics and random matrix theory has recently led to elegant exact calculations of limits and fluctuations, in the work of Baik, Deift, Johansson, and Rains. The homogeneous version of this particular growth model is treated in ref. 15.

The hydrodynamic result of Theorem 2.2 implies a shape result for the last-passage growth model. We will not write down the details of such a conversion, but only note one point: Suppose the diagonal defect does not pass through the origin, but instead through the macroscopic point $(0, u)$ for some $u > 0$. To capture this, redefine the weights in (2.4) as $w_{i,\, i+[nu]} = r^{-1}$, and for other points $w_{i,\, j} \equiv 1$. Now the weights change with n, to keep the defect at the macroscopic line $y = x + u$. The limiting shape for this model follows (2.8) part of the way, but develops a kink around the point where the defect line $y = x + u$ passes through the interface, and the interface is no longer convex. This conclusion can be worked out from the explicit formulas of Section 5.1, via the mapping in Lemma 4.2.

Relation with the Covert–Rezakhanlou result. The definition of $\kappa(r)$ through (2.4)–(2.7) suggests an immediate bound. If we increase all weights in (2.4) to $w_{i,j} = 1/r$ and use Rost's result (2.8), we get the upper bound $\kappa(r) \leqslant 4/r$. By (2.12) this implies $\lambda(0) \geqslant r$, and then from (2.18), $\rho^* \geqslant (1/2) - (1/2)\sqrt{1-r}$. Equivalently, the blockage does not disturb a profile at density ρ if $r \geqslant 4\rho(1-\rho)$. This is the bound of Covert and Rezakhanlou.[4]

To obtain this bound ref. 4 approximated the blocked TASEP with a system whose jump rates depend on spatial location through a continuous function $\tilde{\lambda}(x)$, so that in the nth process jumps from site i occur at rate $\tilde{\lambda}(i/n)$.

The Rost picture shows why a continuous, one-sided *macroscopic* approximation cannot get any closer than the upper bound $\kappa(r) \leqslant 4/r$. Let $\tilde{\lambda}(x, y)$ be a continuous function that satisfies $\tilde{\lambda}(x, y) \leqslant 1$ and $\tilde{\lambda}(x, x) = r$. In (2.5) take new weights $\tilde{w}_{i,j} = \tilde{\lambda}(i/n, j/n)^{-1}$, and compute $\tilde{\kappa}(r)$ as the limit in (2.7) with the weights $\tilde{w}_{i,j}$. Since $w_{i,j} \leqslant \tilde{w}_{i,j}$, we have $\kappa(r) \leqslant \tilde{\kappa}(r)$. Given any $\varepsilon > 0$, choose $\delta > 0$ so that $\tilde{\lambda}(x, y) < r + \varepsilon$ in a strip of width δ around the diagonal $\Delta = \{(x, x): 0 \leqslant x \leqslant 1\}$. In the homogeneous problem (weights $\equiv 1$), the strict concavity of the limit (2.8) implies that, macroscopically, the diagonal is the unique maximizing path for $(x, y) = (1, 1)$. Thus with high probability, the $n\delta$-strip around the microscopic diagonal $\{(i, i): 1 \leqslant i \leqslant n\}$ contains a path π such that $\sum_\pi Y_{i,j} \geqslant 4n + o(n)$. The weights in the $n\delta$-strip satisfy $\tilde{w}_{i,j} \geqslant (r+\varepsilon)^{-1}$, so it follows that $\tilde{\kappa}(r) \geqslant 4(r+\varepsilon)^{-1}$. Since $\varepsilon > 0$ was arbitrary, $\tilde{\kappa}(r) \geqslant 4/r$, and we see that the upper bound $4/r$ is the best we can get by continuously bounding the rates from below.

3. PROOF OF THEOREM 2.1

Let us write $T^r_{n,n}$ for the passage time in (2.5) to indicate dependence on r. Let $0 < r_1 < r_2 \leqslant 1$. Let π be the path that gives the maximum in (2.5) for $T^{r_1}_{n,n}$, so that

$$T^{r_1}_{n,n} = \sum_{\mathbf{u} \in \pi} Y_{\mathbf{u}} + \left(\frac{1}{r_1} - 1\right) \sum_{\mathbf{u} \in \pi \cap \Delta} Y_{\mathbf{u}}$$

We wrote $\mathbf{u} = (i, j)$ for an integer site on the plane and $\Delta = \{(x, x): x \in \mathbf{R}\}$ is the diagonal. Since we are maximizing passage times of paths,

$$T^{r_2}_{n,n} \geqslant \sum_{\mathbf{u} \in \pi} Y_{\mathbf{u}} + \left(\frac{1}{r_2} - 1\right) \sum_{\mathbf{u} \in \pi \cap \Delta} Y_{\mathbf{u}}$$

and we get

$$T^{r_1}_{n,n} - T^{r_2}_{n,n} \leqslant \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \sum_{i=1}^{n} Y_{i,i}$$

Divide by n and let $n \to \infty$ to get the first inequality in Theorem 2.1.

Set $r_2 = 1$ and use $\kappa(1) = 4$ to get the upper bound in (2.9). The value $\kappa(1) = 4$ is part of the lower bound because $\kappa(\cdot)$ is a nondecreasing function.

To get the other part of the lower bound, consider this particular path π_n from $(1, 1)$ to (n, n). For each $i = 1,..., n-1$ choose the larger one of $Y_{i+1,i}$ and $Y_{i,i+1}$ to be included in π_n. Since $E[Y' \vee Y''] = 3/2$ for two independent rate one exponentials, this contributes (approximately) $3n/2$ to the sum. For each pair $\{i, i+1\}$, the above choice led to one of two situations:

Case 1. Two points on the same side of the diagonal: $(i, i+1)$ and $(i+1, i+2)$, or $(i+1, i)$ and $(i+2, i+1)$.

Case 2. Two points on opposite sides of the diagonal: $(i, i+1)$ and $(i+2, i+1)$, or $(i+1, i)$ and $(i+1, i+2)$.

To complete the path π we pick one site between each pair chosen above. In Case 1 we can choose the larger one of the diagonal $r^{-1} Y_{i+1,i+1}$ and an off-diagonal value, $Y_{i,i+2}$ or $Y_{i+2,i}$ depending on the subcase of Case 1. The expected contribution is $E[Y' \vee (r^{-1} Y'')] = (r^2 + r + 1)\, r^{-1}(1+r)^{-1}$. In Case 2 there is only one alternative: the path must go through $(i+1, i+1)$ and take the diagonal value $r^{-1} Y_{i+1,i+1}$. Cases 1 and 2 are equally likely, so this second step contributes approximately $(n/2) \cdot (r^2 + r + 1)\, r^{-1}(1+r)^{-1} + (n/2) \cdot r^{-1}$. Adding the contributions from the two steps gives the lower bound in (2.9).

4. PROOF OF THEOREM 2.2

4.1. Construction of the Process and the Variational Coupling

We follow the construction in Section 4 of ref. 10, and only outline it here. We construct a process $z(t) = (z_i(t): i \in \mathbf{Z})$ of labeled particles that move on $\mathbf{Z}$. The location of the ith particle at time t is $z_i(t)$, and these satisfy

$$0 \leqslant z_{i+1}(t) - z_i(t) \leqslant 1 \tag{4.1}$$

For the dynamics, let $\{\mathcal{D}_i\}$ be a collection of mutually independent Poisson jump time processes on the time line $(0, \infty)$. $\mathcal{D}_0$ has rate r, and all other $\mathcal{D}_i$ have rate 1. In the graphical construction, z_i attempts a jump one step to the *left* at epochs of $\mathscr{D}_i$. The jump is executed if it does not violate (4.1). This happens independently for all i.

Once the $z(t)$ process is constructed, the exclusion process $\eta(t)$ is defined by

$$\eta_i(t) = z_i(t) - z_{i-1}(t) \tag{4.2}$$

The z-particles keep track of the current of the exclusion process $\eta(\cdot)$. The number of η-particles that have left site i in time $[0, t]$ is given by

$$J_i(t) = z_i(0) - z_i(t) \tag{4.3}$$

Assume now that the process $z(\cdot)$ has been constructed on some probability space that supports the initial configuration $z(0) = (z_i(0))$, and the Poisson processes $\{\mathscr{D}_i\}$ that are independent of $(z_i(0))$. We define a family $\{w^k: k \in \mathbf{Z}\}$ of auxiliary processes on this same probability space. Each $w^k(t) = (w_i^k(t): i \in \mathbf{Z})$ is an exclusion process just like $z(t)$, so (4.1) is in force for all i. Initially

$$w_i^k(0) = \begin{cases} z_k(0), & i \geqslant 0 \\ z_k(0) + i, & i < 0 \end{cases} \tag{4.4}$$

and dynamically

$$w_i^k \text{ attempts to jump to } w_i^k - 1 \text{ at the epochs of } \mathscr{D}_{i+k} \tag{4.5}$$

The usefulness of the family of processes $\{w^k\}$ lies in this fact:

Lemma 4.1. For all $i \in \mathbf{Z}$ and $t \geqslant 0$,

$$z_i(t) = \sup_{k \in \mathbf{Z}} w_{i-k}^k(t) \qquad \text{a.s.} \tag{4.6}$$

This lemma is proved as Lemma 4.2 in ref. 10 so we will not repeat the proof here. It is the "variational coupling" that is the key to our proof.

Thinking of $w^k(t)$ as the height of an interface over the sites i, we normalize it to start at height zero and to advance in the increasing coordinate direction. To this end define a new family of processes $\{\xi^k\}$ by

$$\xi_i^k(t) = z_k(0) - w_i^k(t) \qquad \text{for} \quad i \in \mathbf{Z}, \quad t \geqslant 0 \tag{4.7}$$

Now we can write (4.6) as

$$z_i(t) = \sup_{k \in \mathbf{Z}} \{z_k(0) - \xi_{i-k}^k(t)\} \tag{4.8}$$

The virtue of (4.8) is that the effect of the initial condition $(z_k(0))$ has been separated from the effect of the Poisson jump times $\{\mathcal{D}_i\}$. The process ξ^k does not depend on $z_k(0)$, and depends on the superscript k only through a translation of the indexing of $\{\mathcal{D}_i\}$. Initially

$$\xi_i^k(0) = \begin{cases} 0, & i \geqslant 0 \\ -i, & i < 0 \end{cases} \tag{4.9}$$

The dynamical rule for the ξ^k process is that

$$\xi_i^k \text{ jumps to } \xi_i^k + 1 \text{ at epochs of } \mathcal{D}_{i+k} \tag{4.10}$$

provided the inequalities

$$\xi_i^k \leqslant \xi_{i-1}^k \qquad \text{and} \qquad \xi_i^k \leqslant \xi_{i+1}^k + 1 \tag{4.11}$$

are not violated. Notice that ξ_{-k}^k jumps at rate r, while other ξ_i^k jump at rate 1.

We can now outline the strategy for proving Theorem 2.2. Given the initial configurations $\eta^n(0) = (\eta_i^n(0) \colon i \in \mathbf{Z})$ that appear in hypothesis (2.16), define initial configurations $z^n(0) = (z_i^n(0) \colon i \in \mathbf{Z})$ so that $z_0^n(0) = 0$ and (4.2) holds at time $t = 0$. Then hypothesis (2.16) implies that

$$\lim_{n \to \infty} n^{-1} z_{[nq]}^n(0) = v_0(q) \qquad \text{in probability} \tag{4.12}$$

for all $q \in \mathbf{R}$, with v_0 defined by (2.11).

Construct the processes $z^n(t)$ as indicated above, and define the exclusion processes $\eta^n(t)$ by (4.2). Define $v(x, t)$ by (2.14). From (4.2)–(4.3) we see that both limits of Theorem 2.2 follow from proving that for all $x \in \mathbf{R}$ and $t > 0$,

$$\lim_{n \to \infty} n^{-1} z_{[nx]}^n(nt) = v(x, t) \qquad \text{in probability} \tag{4.13}$$

Now rewrite (4.8) with the correct scaling:

$$n^{-1} z_{[nx]}^n(nt) = \sup_{q \in \mathbf{R}} \{ n^{-1} z_{[nq]}^n(0) - n^{-1} \xi_{[nx]-[nq]}^{[nq]}(nt) \} \tag{4.14}$$

In this formula each process $z^n(\cdot)$ is defined on a probability space that supports the initial configuration $z^n(0)$ and the Poisson processes $\{\mathcal{D}_i\}$. On each such probability space we define the processes $\{\xi^k(\cdot)\}$ as functions of $\{\mathcal{D}_i\}$, according to (4.9)–(4.11). The proof of (4.13) is now to show that the right-hand side of (4.14) converges to the right-hand side of (2.14). The first step is the limit for the ξ-term.

4.2. Limit for ξ

First some definitions. The meaning of these notions will be explained below. Let

$$\mathscr{V} = \{(x, y) \in \mathbf{R}^2 : y \geqslant 0, x \geqslant -y\}$$

For $(x, y) \in \mathscr{V}$, let

$$\gamma_0(x, y) = (\sqrt{x+y} + \sqrt{y}\,)^2 \tag{4.15}$$

Let $\mathbf{x}(s) = (x_1(s), x_2(s))$ denote a path in $\mathbf{R}^2$, defined on some interval of s-values. For $(x, y) \in \mathscr{V}$ and $q \in \mathbf{R}$ let

$$\Gamma^q(x, y) = \sup \left\{ \int_0^1 \frac{\gamma_0(\mathbf{x}'(s))}{\lambda(x_1(s) - q)}\, ds : \mathbf{x}(\cdot) \in \mathscr{X}(x, y) \right\} \tag{4.16}$$

where $\mathscr{X}(x, y)$ is the collection of piecewise C^1 paths $\mathbf{x}$: $[0, 1] \to \mathscr{V}$ that satisfy

$$\mathbf{x}(0) = (0, 0), \quad \mathbf{x}(1) = (x, y), \quad \text{and} \quad \mathbf{x}'(s) \in \mathscr{V} \qquad \text{for all } s \tag{4.17}$$

The last condition ensures that $\gamma_0(\mathbf{x}'(s))$ is defined. The function $\lambda(\cdot)$ in the definition of Γ^q is the macroscopic rate defined by (2.12). Lastly, for $q, x \in \mathbf{R}$ and $t > 0$, set

$$g^q(x, t) = \inf\{ y \colon (x, y) \in \mathscr{V}, \Gamma^q(x, y) \geqslant t\} \tag{4.18}$$

$\Gamma^q(x, y)$ represents the macroscopic time it takes a ξ-type interface process to reach point (x, y). The point q marks the x-coordinate of the defect column that (macroscopically) has the slow rate $\lambda(0)$. The level curve of Γ^q given by $g^q(\cdot, t)$ represents the limiting interface of a ξ-process, as stated in the next proposition.

Proposition 4.1. For all $q, x \in \mathbf{R}$ and $t > 0$,

$$\lim_{n \to \infty} n^{-1} \xi^{[nq]}_{[nx]}(nt) = g^{-q}(x, t) \qquad \text{in probability} \tag{4.19}$$

Remark 4.1. For the reader familiar with the proof of ref. 10, let us point out that Proposition 4.1 is the analogue of Corollary 5.1 in ref. 10, and Γ^q corresponds to the limit in Proposition 5.1 in ref. 10.

To prove Proposition 4.1 we follow Section 5 of ref. 10 and switch to a last-passage representation. Lemma 4.2 below shows how the growth model of Section 2.1 enters the picture, and justifies (2.7) as the definition of $\kappa(r)$.

Define a lattice analogue of the interior of the wedge $\mathscr{V}$ by $\mathscr{L} = \{(i, j) \in \mathbf{Z}^2 : j \geqslant 1, i \geqslant -j+1\}$, with boundary $\partial\mathscr{L} = \{(i,0): i \geqslant 0\} \cup \{(i, -i): i < 0\}$. For $(i, j) \in \mathscr{L} \cup \partial\mathscr{L}$, let

$$L^k(i, j) = \inf\{t \geqslant 0 : \xi_i^k(t) \geqslant j\} \tag{4.20}$$

denote the time when ξ_i^k reaches level j. The rules (4.9)–(4.11) give the boundary conditions

$$L^k(i, j) = 0 \qquad \text{for} \quad (i, j) \in \partial\mathscr{L} \tag{4.21}$$

and for $(i, j) \in \mathscr{L}$ the equation

$$L^k(i, j) = \max\{L^k(i-1, j), L^k(i, j-1), L^k(i+1, j-1)\} + \beta^k_{i,j} \tag{4.22}$$

where $\beta^k_{i,j}$ is an exponential waiting time, independent of everything else. It represents the time ξ_i^k waits to jump, *after* ξ_i^k and its neighbors ξ^k_{i-1}, ξ^k_{i+1} have reached the positions that permit ξ_i^k to jump to j. By (4.10), $\beta^k_{-k,j}$ has rate r, but for $i \neq -k$, $\beta^k_{i,j}$ has rate 1.

The waiting time $\beta^k_{i,j}$ cannot be read directly from $\mathscr{D}_{i+k}$. One has to construct the evolution $\xi^k(\cdot)$ up to the stopping time $L = \max\{L^k(i-1, j), L^k(i, j-1), L^k(i+1, j-1)\}$, and then $\beta^k_{i,j}$ is the waiting time to the next epoch in $\mathscr{D}_{i+k}$ after L. The last-passage representation entails switching probability spaces so that the waiting times $\beta^k_{i,j}$ become the basic building blocks of the construction. We shall switch notation to keep the two constructions distinct.

We now construct a last-passage growth model on $\mathscr{L}$ that has a defect in the column $\{(m, j): j \in \mathbf{Z}\}$, where $m \in \mathbf{Z}$ is fixed. Let $\{\tau_{\mathbf{u}}: \mathbf{u} \in \mathscr{L}\}$ denote a collection of i.i.d. exponential rate 1 random variables. Define weights

$$\omega^m_{i,j} = \begin{cases} 1, & i \neq m \\ 1/r, & i = m \end{cases} \tag{4.23}$$

Given $\mathbf{u} = (u, v) \in \mathscr{L}$, let $\Pi(\mathbf{u})$ denote the set of lattice paths $\pi = \{(0, 1) = (i_1, j_1), (i_2, j_2),..., (i_p, j_p) = \mathbf{u}\}$ whose admissible steps satisfy

$$(i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1}) = (1, 0), (0, 1), \qquad \text{or} \qquad (-1, 1) \tag{4.24}$$

Finally, define the passage times $\{T^m(\mathbf{u})\}$ by

$$T^m(\mathbf{u}) = 0 \qquad \text{for} \quad \mathbf{u} \in \partial\mathscr{L} \tag{4.25}$$

and for $\mathbf{u} \in \mathscr{L}$ as the maximal weighted sum of waiting times over admissible paths:

$$T^m(\mathbf{u}) = \max_{\pi \in \Pi(\mathbf{u})} \sum_{\mathbf{v} \in \pi} \omega_{\mathbf{v}}^m \tau_{\mathbf{v}} \tag{4.26}$$

Comparison of (4.21)–(4.22) and (4.25)–(4.26) shows that the passage time processes $\{L^k(\mathbf{u}): \mathbf{u} \in \mathscr{L} \cup \partial \mathscr{L}\}$ and $\{T^{-k}(\mathbf{u}): \mathbf{u} \in \mathscr{L} \cup \partial \mathscr{L}\}$ have the same distribution. The superscript changes by a minus sign from $L^k(\mathbf{u})$ to $T^{-k}(\mathbf{u})$ because $L^k(\mathbf{u})$ is the passage time for the process ξ^k, and the slow column for this process is at $-k$, so we set $m = -k$ in $T^m(\mathbf{u})$.

The conclusion of this is that $T^{-k}(i, j)$ has the same distribution as the time when ξ_i^k reaches level j. So, as in ref. 10, Proposition 4.1 will follow if we prove

Proposition 4.2. For all $q \in \mathbf{R}$ and (x, y) in the interior of $\mathscr{V}$,

$$\lim_{n \to \infty} n^{-1} T^{[nq]}([nx], [ny]) = \Gamma^q(x, y) \quad \text{in probability} \tag{4.27}$$

Remark 4.2. Let us first explain the known homogeneous situation where $r = 1$ and $\lambda(0) = 1$. Write $\xi^{(r=1)}$ and $T^{(r=1)}$ for the interface process defined by (4.9)–(4.11) and the passage time in (4.26) when $r = 1$ and there is no special column. In this case the limits in (4.19) and (4.27) are given by

$$\lim_{n \to \infty} n^{-1} \xi_{[nx]}^{(r=1)}(nt) = t g_0(x/t) = \frac{t}{4}\left(1 - \frac{x}{t}\right)^2 \quad \text{for} \quad -t \leqslant x \leqslant t, \quad \text{and}$$
$$\lim_{n \to \infty} n^{-1} T^{(r=1)}([nx], [ny]) = \gamma_0(x, y) = (\sqrt{x + y} + \sqrt{y}\,)^2 \tag{4.28}$$

The explicit values cannot be inferred directly from the path model, but indirectly via the connection with TASEP. Briefly, one computes the current $f_0(\rho) = \rho(1 - \rho)$ from the known invariant distributions of TASEP. The coupling (4.8) implies that the limiting shape g_0 for ξ is the conjugate of f_0, and one can derive g_0. The shape g_0 is a level curve of the passage time γ_0, so one obtains the formula for γ_0 from

$$\gamma_0(x, g_0(x)) = 1 \tag{4.29}$$

and the homogeneity of γ_0. This type of argument is repeated in the examples in ref. 3. An alternative way to compute these explicit values is the random matrix approach of ref. 15.

Proof of Proposition 4.2. Consider the possible maximizing macroscopic curves in (4.16). By the concavity of γ_0, of all the paths $\mathbf{x}(s) = (x_1(s), x_2(s))$ in $\mathscr{X}(x, y)$, we only need to consider these two types:

$$\mathbf{x}(s) = (sx, sy), \qquad 0 \leqslant s \leqslant 1 \tag{4.30}$$

with value [except in the case $x = q = 0$ which is covered by (4.32)]

$$\int_0^1 \frac{\gamma_0(\mathbf{x}'(s))}{\lambda(x_1(s) - q)} \, ds = \gamma_0(x, y) \tag{4.31}$$

and

$$\mathbf{x}(s) = \begin{cases} \dfrac{s}{s_1}(q, x_2(s_1)), & 0 \leqslant s \leqslant s_1 \\[2ex] (q, x_2(s_1)) + \dfrac{s - s_1}{s_2 - s_1}(0, x_2(s_2) - x_2(s_1)), & s_1 \leqslant s \leqslant s_2 \\[2ex] (q, x_2(s_2)) + \dfrac{s - s_2}{1 - s_2}(x - q, y - x_2(s_2)), & s_2 \leqslant s \leqslant 1 \end{cases} \tag{4.32}$$

with value

$$\int_0^1 \frac{\gamma_0(\mathbf{x}'(s))}{\lambda(x_1(s) - q)} \, ds = \gamma_0(\mathbf{x}(s_1)) + \lambda(0)^{-1} \gamma_0(\mathbf{x}(s_2) - \mathbf{x}(s_1)) + \gamma_0((x, y) - \mathbf{x}(s_2)) \tag{4.33}$$

In words: the path of type (4.30) is a single line segment from $(0, 0)$ to (x, y). The path of type (4.32) first uses parameter interval $[0, s_1]$ to take a straight line path to the vertical line $x = q$, then spends interval $[s_1, s_2]$ on this line to take advantage of the slow rate $\lambda(0)$, and finally takes a straight line path to (x, y).

Now consider the microscopic path problem (4.26). We need to establish the connection between it and the quantity $\lambda(0) = 4/\kappa(r)$ defined in Section 2.1.

Lemma 4.2. Set $q = 0$ so that the special column goes through the origin. Then for $y > 0$

$$\lim_{n \to \infty} n^{-1} T^0(0, [ny]) = \kappa(r) y \qquad \text{in probability} \tag{4.34}$$

Proof of Lemma 4.2. The growth model of Section 2.1 with a diagonal defect is the same as the one studied here when the columnar defect is at the origin. A simple mapping reveals this. First observe that the admissible step $(0, 1)$ can be eliminated from (4.24), because each $(0, 1)$-step in a path can be replaced by a $(1, 0)$-step followed by a $(-1, 1)$-step. This change adds a site to the path and hence increases its overall passage time. So we may assume that $\Pi(\mathbf{u})$ contains only paths that have admissible steps $(1, 0)$ and $(-1, 1)$.

Consider the bijection $\psi: \mathscr{L} \to \mathbf{N}^2$ given by $\psi(i, j) = (i+j, j)$. For $(i, j) \in \mathbf{N}^2$, define random variables $Y_{i,j} = \tau_{\psi^{-1}(i,j)}$ and weights $w_{i,j} = \omega^0_{\psi^{-1}(i,j)}$. Then $w_{i,j}$ satisfies (2.4). For $\pi \in \Pi(0, n)$, the image path $\psi(\pi)$ runs from $(1, 1)$ to (n, n), and has steps of two kind: $(1, 0)$ and $(0, 1)$. Thus the map ψ transforms $T^0(0, n)$ of (4.26) into $T_{n,n}$ of (2.5). Now Lemma 4.2 follows from the definition (2.7) of $\kappa(r)$. ∎

Return to the proof of Proposition 4.2. To first prove

$$\liminf_{n \to \infty} n^{-1} T^{[nq]}([nx], [ny]) \geqslant \Gamma^q(x, y) \tag{4.35}$$

consider any macroscopic path $\mathbf{x}(\cdot)$ of type (4.32). [We leave the easier type (4.30) to the reader.] Let π_n be the microscopic path through the four sites

$$(0, 1), \ [n\mathbf{x}(s_1)] \equiv ([nx_1(s_1)], [nx_2(s_1)]), [n\mathbf{x}(s_2)], \text{ and } ([nx], [ny])$$

constructed so that each of the three segments maximizes passage time between its endpoints. Then by (4.34) and the limit (4.28) for the homogeneous case,

$$\begin{aligned}
&\liminf_{n \to \infty} n^{-1} T^{[nq]}([nx], [ny]) \\
&\qquad \geqslant \liminf_{n \to \infty} n^{-1} \sum_{\mathbf{v} \in \pi_n} \omega^{[nq]}_{\mathbf{v}} \tau_{\mathbf{v}} \\
&\qquad \geqslant \gamma_0(\mathbf{x}(s_1)) + \kappa(r)(x_2(s_2) - x_2(s_1)) + \gamma_0((x, y) - \mathbf{x}(s_2)) \\
&\qquad = \text{the value of the path in (4.33)}
\end{aligned} \tag{4.36}$$

The last equality follows because

$$\begin{aligned}
\lambda(0)^{-1} \gamma_0(\mathbf{x}(s_2) - \mathbf{x}(s_1)) &= (1/4)\, \kappa(r)\, \gamma_0(0, x_2(s_2) - x_2(s_1)) \\
&= \kappa(r)(x_2(s_2) - x_2(s_1))
\end{aligned}$$

The reason there might not be equality in the last inequality in (4.36) is that an optimal path between, say, $(0, 1)$ and $[n\mathbf{x}(s_1)]$ might actually take advantage of the $[nq]$-column and return a larger value than $\gamma_0(\mathbf{x}(s_1))$. A similar argument for paths of type (4.30) justifies (4.35).

Now for the complementary upper bound

$$\limsup_{n\to\infty} n^{-1} T^{[nq]}([nx], [ny]) \leqslant \Gamma^q(x, y) \tag{4.37}$$

Each macroscopic path $\mathbf{x}(\cdot)$ in $\mathscr{X}(x, y)$ is contained in a fixed compact subset A of $\mathscr{V}$. Choose $\delta > 0$ so that

$$|\gamma_0(\mathbf{x}) - \gamma_0(\mathbf{y})| < \varepsilon \quad \text{for } \mathbf{x}, \mathbf{y} \in A \quad \text{such that } |\mathbf{x} - \mathbf{y}| < \delta \tag{4.38}$$

and then a partition

$$0 = b_0 < b_1 < \cdots < b_s = y$$

of $[0, y]$ with mesh $\max(b_{i+1} - b_i) < \delta$. Let $\mathbf{x}^{(i, j)}$ be the path of type (4.32) with $x_2^{(i, j)}(s_1) = b_i < b_j = x_2^{(i, j)}(s_2)$.

Let us adopt the following generalization of the notation in (4.26): $T^m(\mathbf{u}, \mathbf{v})$ denotes the maximal weighted sum over admissible paths from $\mathbf{u}$ to $\mathbf{v}$. So $T^m(\mathbf{u})$ in (4.26) is the same as $T^m((0, 1), \mathbf{u})$.

Let π_n be the maximizing microscopic path in (4.26) for $\mathbf{u} = ([nx], [ny])$ and $m = [nq]$. The easy situation is when π_n does not intersect the vertical column $([nq], j)$ that has the slow rate r. Then $T^{[nq]}([nx], [ny])$ equals the homogeneous passage time $T^{(r=1)}([nx], [ny])$. If this happens infinitely often along the subsequence taken on the left-hand side of (4.37), then (4.37) follows from (4.28).

Otherwise, pick indices $k \leqslant \ell$ such that the path π_n first touches the vertical column $([nq], j)$ in the range $[nb_k] \leqslant j \leqslant [nb_{k+1}]$, and for the last time in the range $[nb_\ell] \leqslant j \leqslant [nb_{\ell+1}]$. Then quite obviously

$$\begin{aligned}
&T^{[nq]}([nx], [ny]) \\
&\quad = \sum_{\mathbf{v} \in \pi_n} \omega_{\mathbf{v}}^{[nq]} \tau_{\mathbf{v}} \\
&\quad \leqslant T^{(r=1)}([nq], [nb_{k+1}]) + T^{[nq]}(([nq], [nb_k])([nq], [nb_{\ell+1}])) \\
&\qquad + T^{(r=1)}(([nq], [nb_\ell]), ([nx], [ny])) \\
&\quad \leqslant \max_{i \leqslant j} \{T^{(r=1)}([nq], [nb_{i+1}]) + T^{[nq]}(([nq], [nb_i]), ([nq], [nb_{j+1}])) \\
&\qquad + T^{(r=1)}(([nq], [nb_j]), ([nx], [ny]))\}
\end{aligned}$$

Divide by n, let $n \to \infty$, use the limits (4.28) and (4.34), and then (4.38) to get, with $C = 1 + \lambda(0)^{-1}$,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-1} T^{[nq]}([nx], [ny]) \\
&\quad \leqslant \max_{i \leqslant j} \{\gamma_0(q, b_{i+1}) + \kappa(r)(b_{j+1} - b_i) + \gamma_0(x-q, y-b_j)\} \\
&\quad \leqslant \max_{i \leqslant j} \{\gamma_0(q, b_i) + \lambda(0)^{-1}\, \gamma_0(0, b_j - b_i) + \gamma_0(x-q, y-b_j)\} + C\varepsilon \\
&\quad = \max_{i \leqslant j} \{\text{value of path } \mathbf{x}^{(i,j)}(\cdot)\} + C\varepsilon \\
&\quad \leqslant \Gamma^q(x, y) + C\varepsilon
\end{aligned}$$

This completes the proof of Proposition 4.2. ▮

Proposition 4.1 follows from Proposition 4.2 as Corollary 5.1 follows from Proposition 5.1 in ref. 10.

4.3. Hydrodynamic Limit

Using (4.12), (4.14), and (4.19) we can now prove that, in probability,

$$\lim_{n\to\infty} n^{-1} z^n_{[nx]}(nt) = \tilde{v}(x, t) \equiv \sup_{q \in \mathbf{R}} \{v_0(q) - g^{-q}(x-q, t)\} \tag{4.39}$$

The argument is the one from Eqs. (6.4) to (6.15) in ref. 10, so we will not repeat it here. To complete the proof of (4.13) and thereby the proof of Theorem 2.2, we need to show that the limiting value $\tilde{v}(x, t)$ defined above agrees with the desired limit $v(x, t)$ defined by (2.14).

4.4. Formula for *v*(*x*, *t*)

By (4.16)–(4.18), the definition (4.39) of $\tilde{v}(x, t)$ can be rewritten as

$$\tilde{v}(x, t) = \sup_{q,\, y \in \mathbf{R}} \left\{ v_0(q) - y\text{: there exists a path } \mathbf{x}(\cdot) \in \mathscr{X}(x-q, y) \right.$$
$$\left. \text{such that } \int_0^1 \frac{\gamma_0(\mathbf{x}'(s))}{\lambda(x_1(s)+q)}\, ds \geqslant t \right\} \tag{4.40}$$

Proposition 4.3. $v(x, t) = \tilde{v}(x, t)$.

Proof. The proof involves mapping the paths $\mathbf{x}(\cdot)$ in (4.40) to the paths $w(\cdot)$ in (2.14), and vice versa.

Given a path $\mathbf{x}(\cdot) \in \mathcal{X}(x-q, y)$ that appears in (4.40), define a new time variable $\tau = \tau(s)$ by

$$\tau(s) = \int_0^s \frac{\gamma_0(\mathbf{x}'(s))}{\lambda(x_1(s)+q)}\, ds \tag{4.41}$$

Let the terminal τ-time be $t_1 = \tau(1)$. From (4.40) we know that $t_1 \geqslant t$. Let $s = s(\tau)$ be the inverse time change. Define a path z: $[0, t_1] \to \mathbf{R}$ by

$$z(\tau) = q + x_1(s(\tau)) \tag{4.42}$$

Then

$$z(0) = q, \quad z(t_1) = x, \quad \text{and} \quad z'(\tau) = x_1'(s(\tau))\, s'(\tau) \tag{4.43}$$

Differentiating (4.41), relation (4.29), and the homogeneity of γ_0 [means: $\gamma_0(cx, cy) = c\gamma_0(x, y)$] give

$$x_2'(s) = \tau'(s)\, \lambda(x_1(s)+q)\, g_0\left(\frac{x_1'(s)}{\tau'(s)\, \lambda(x_1(s)+q)}\right) \tag{4.44}$$

Since $y = x_2(1)$, we can use this to compute

$$\begin{aligned} y &= \int_0^1 x_2'(s)\, ds = \int_0^1 \tau'(s)\, \lambda(x_1(s)+q)\, g_0\left(\frac{x_1'(s)}{\tau'(s)\, \lambda(x_1(s)+q)}\right) ds \\ &= \int_0^{t_1} \lambda(z(\tau))\, g_0\left(\frac{z'(\tau)}{\lambda(z(\tau))}\right) d\tau \end{aligned}$$

The only problem is that $z(\cdot)$ is defined on $[0, t_1]$ instead of on the possibly smaller interval $[0, t]$. Let $w(\sigma) = z(t_1 \sigma / t)$ be a time change of $z(\cdot)$ defined for $0 \leqslant \sigma \leqslant t$. Then, because $s g_0(x/s)$ is nondecreasing in s, change of variable $\tau = (t_1/t)\, \sigma$ in the last integral above shows that

$$v_0(q) - y \leqslant v_0(w(0)) - \int_0^t \lambda(w(\sigma))\, g_0\left(\frac{w'(\sigma)}{\lambda(w(\sigma))}\right) d\sigma$$

Since $\mathbf{x}(\cdot)$ was an arbitrary path inside the braces in (4.40), we have shown that $\tilde{v}(x, t) \leqslant v(x, t)$.

Conversely, take a path w: $[0, t] \to \mathbf{R}$ that appears in (2.14). Define a path $\mathbf{x}(\sigma) = (x_1(\sigma), x_2(\sigma))$ for $\sigma \in [0, 1]$ by

$$\begin{aligned} x_1(\sigma) &= w(\sigma t) - w(0) \\ x_2(\sigma) &= \int_0^{\sigma t} \lambda(w(s))\, g_0\left(\frac{w'(s)}{\lambda(w(s))}\right) ds \end{aligned} \tag{4.45}$$

Let $q = w(0)$ and $y = x_2(1)$. Then $\mathbf{x}(\cdot) \in \mathscr{X}(x-q, y)$, provided $\mathbf{x}'(\sigma) \in \mathscr{V}$ [recall conditions (4.17)]. This follows because λ and g_0 are nonnegative functions, and because $g(z) \geqslant -z$ for all z. Also, (4.29) and (4.45) give

$$\frac{\gamma_0(\mathbf{x}'(\sigma))}{\lambda(x_1(\sigma)+q)} = t$$

so the integral condition inside the braces in (4.40) is satisfied. We conclude that $\mathbf{x}(\cdot)$ is a path that appears in (4.40), and since

$$v_0(w(0)) - \int_0^t \lambda(w(s))\, g_0\left(\frac{w'(s)}{\lambda(w(s))}\right) ds = v_0(q) - y \leqslant \tilde{v}(x, t)$$

we have $v(x, t) \leqslant \tilde{v}(x, t)$. This completes the proof of Proposition 4.3. ▮

We have now proved Theorem 2.2.

5. PROOFS OF THE COROLLARIES

5.1. Proof of Corollary 2.1

As a preliminary step for calculating macroscopic profiles from the variational formula (2.14), we optimize the integral term as a function of the initial point $q = w(0)$. So let

$$I(x, t, q) = \inf \left\{ \int_0^t \lambda(w(s))\, g_0\left(\frac{w'(s)}{\lambda(w(s))}\right) ds : w\colon [0, t] \to \mathbf{R} \text{ is piecewise } C^1,\ w(0) = q, \text{ and } w(t) = x \right\} \tag{5.1}$$

In terms of the limiting shapes of Proposition 4.1, $I(x, t, q) = g^{-q}(x-q, t)$, so from the formulas below the reader can deduce expressions for the limits in (4.19).

By the convexity of g_0, it suffices to consider the following two types of paths in (5.1): either $w(\cdot)$ is a single linear segment from $q = w(0)$ to $w(t) = x$; or it consists of a linear segment from $q = w(0)$ to $w(s_1) = 0$, a constant segment $w(s) = 0$ for $s_1 \leqslant s \leqslant s_2$, and a linear segment from $w(s_2) = 0$ to $w(t) = x$. Only calculus is involved in finding the optimal paths, so we skip the details and present a summary of the results. Abbreviate

$$B = \sqrt{1 - \lambda(0)}$$

Five different ranges of the variables x and q appear.

$$|x| \geqslant Bt \tag{5.2a}$$

$$0 \leqslant x < Bt, \quad \text{and} \quad q \leqslant x - Bt \quad \text{or} \quad q \geqslant (\sqrt{Bt} - \sqrt{x}\,)^2 \tag{5.2b}$$

$$-Bt < x \leqslant 0, \quad \text{and} \quad q \leqslant -(\sqrt{Bt} - \sqrt{|x|}\,)^2 \quad \text{or} \quad q \geqslant x + Bt \tag{5.2c}$$

$$0 \leqslant x < Bt, \quad \text{and} \quad x - Bt < q < (\sqrt{Bt} - \sqrt{x}\,)^2 \tag{5.2d}$$

$$-Bt < x \leqslant 0, \quad \text{and} \quad -(\sqrt{Bt} - \sqrt{|x|}\,)^2 < q < x + Bt \tag{5.2e}$$

These are the optimal values:

Cases (5.2a–c): $I(x, t, q) = tg_0((x-q)/t)$ and the optimal path is $w(s) = q + (s/t)(x-q)$.

Cases (5.2d–e):

$$I(x, t, q) = \frac{|q|}{B}\, g_0(-Bq/|q|) + \left(t - \frac{|x| + |q|}{B}\right) \lambda(0)\, g_0(0) + \frac{|x|}{B}\, g_0(Bx/|x|)$$

$$= \frac{B}{2}\,(|x| + |q|) - \frac{x-q}{2} + \frac{t}{4}\,\lambda(0)$$

and the optimal path is

$$w(s) = \begin{cases} q - sq/s_1, & 0 \leqslant s < s_1 \\ 0, & s_1 \leqslant s < s_2 \\ (s - s_2)\, x/(1 - s_2), & s_2 \leqslant s \leqslant 1 \end{cases}$$

with

$$s_1 = |q|/B \qquad \text{and} \qquad s_2 = t - |x|/B$$

Proof of Corollary 2.1 is now reduced to finding

$$v(x, t) = \sup_{q \in \mathbf{R}} \{v_0(q) - I(x, t, q)\} \tag{5.3}$$

with $v_0(q) = \rho q$. We skip the calculus details.

5.2. Proof of Corollary 2.2

Assume now that $\rho^* < 1/2$, in other words, that the slow bond disturbs the hydrodynamic profiles. First check from (5.3) that all the profiles admitted by the restrictions stated in Corollary 2.2 are in fact invariant.

This contains the following cases: constants at ρ^* and at $1-\rho^*$; a piecewise constant profile with a single entropy shock [jump from ρ^* to $1-\rho^*$] anywhere in $\mathbf{R}$; a piecewise constant profile with a single non-entropy shock [jump from $1-\rho^*$ to ρ^*] at $x=0$; and a piecewise constant profile with a non-entropy shock at $x=0$, and an entropy shock in $(-\infty, 0)$, or in $(0, \infty)$, or in both.

To prove that these are the only possible invariant profiles in the range $[\rho^*, 1-\rho^*]$, let ρ_0 be a profile such that $\rho^* \leqslant \rho_0(x) \leqslant 1-\rho^*$ and ρ_0 is invariant under (5.3). Define v_0 by (2.11). The invariance means that $v_x(x, t) = \rho_0(x)$ a.e., and so

$$v(x, t) = \int_0^x \rho_0(y)\, dy + v(0, t) = v_0(x) + v(0, t) \tag{5.4}$$

Formula (5.3) operates like a semigroup, and this with (5.4) implies that $v(0, t) = Ct$ for some constant C. To determine C, we do a comparison. Let

$$v_0^e(x) = \rho^* x \mathbf{1}\{x<0\} + (1-\rho^*)\, x \mathbf{1}\{x>0\}, \qquad v^e(x, t) = v_0^e(x) - t\lambda(0)/4$$

and

$$v_0^n(x) = (1-\rho^*)\, x \mathbf{1}\{x<0\} + \rho^* x \mathbf{1}\{x>0\}, \qquad v^n(x, t) = v_0^n(x) - t\lambda(0)/4$$

denote the evolution of the entropy and the non-entropy shock at the origin. The bounds $v^n(x, t) \leqslant v(x, t) \leqslant v^e(x, t)$ are valid because they are valid at time $t=0$ and preserved by (5.3). Consequently

$$v(x, t) = v_0(x) - t\lambda(0)/4 \tag{5.5}$$

Lemma 5.1. There cannot exist an $x \neq 0$, and $\varepsilon, \delta > 0$ such that this holds:

$$\begin{aligned} &v_0(x) \geqslant v_0(q) + (\rho^* + \delta)(x-q) && \text{for } q \in [x-\varepsilon, x] \quad \text{and} \\ &v_0(x) \geqslant v_0(q) - (1-\rho^*-\delta)(q-x) && \text{for } q \in [x, x+\varepsilon] \end{aligned} \tag{5.6}$$

Proof. Suppose such x, ε, δ exist. Pick $t>0$ small enough so that $t<\varepsilon$ and $Bt<|x|$. Then, by (5.2), (5.3) becomes

$$v(x, t) = \sup_{q \in \mathbf{R}} \{v_0(q) - t g_0((x-q)/t)\} \tag{5.7}$$

We shall show that formula (5.7) gives something strictly smaller than (5.5), and this contradiction makes (5.6) impossible.

In (5.7) it suffices to consider $q \in [x-t, x+t]$, by observing from (2.13) that g_0 has constant slopes to the left of -1 and to the right of 1. For $q \in [x, x+t]$ write

$$v_0(q) - tg_0((x-q)/t) \leqslant v_0(x) + (1-\rho^*-\delta)(q-x) - tg_0((x-q)/t)$$
$$= v_0(x) - t\{(1-\rho^*-\delta)\,\xi + g_0(\xi)\}$$

where $\xi = (x-q)/t$ may vary freely in $[-1, 0]$. By the duality of g_0 and the TASEP current $f_0(\rho) = \rho(1-\rho)$, the expression in braces is bounded below by $f_0(1-\rho^*-\delta) = f_0(\rho^*+\delta)$. This and a similar argument for $q \in [x-t, x]$ give

$$v(x, t) \leqslant v_0(x) - tf_0(\rho^*+\delta)$$

Since $\lambda(0)/4 = f_0(\rho^*)$, the above bound is strictly less than (5.5), provided δ is chosen small enough to have $\rho^*+\delta < 1/2$. ∎

Now we can prove that ρ_0 takes only the values $\{\rho^*, 1-\rho^*\}$, up to Lebesgue null sets. For suppose the set $A = \{x\colon \rho^*+\delta_0 \leqslant \rho_0(x) \leqslant 1-\rho^*-\delta_0\}$ has positive Lebesgue measure for some $\delta_0 > 0$. Then A has a density point x. (For a definition, see for example p. 107 in ref. 16.) We show that (5.6) holds at x. Let $q < x$. Since $\rho_0(x) \geqslant \rho^*$,

$$v_0(x) - v_0(q) = \int_{[q,x] \cap A} \rho_0(y)\,dy + \int_{[q,x] \cap A^c} \rho_0(y)\,dy$$
$$\geqslant \rho^*(x-q) + \delta_0 \cdot m([q, x] \cap A)$$

where we wrote m for Lebesgue measure. Since x is a density point, $m([q, x] \cap A) \geqslant (x-q)/2$ if q is close enough to x. This checks the first part of (5.6), and the other part is similar.

Assuming that ρ_0 is piecewise continuous, we now conclude that it is piecewise constant with values ρ^* and $1-\rho^*$. To prove Corollary 2.2, it remains to observe that Lemma 5.1 prevents a jump from $1-\rho^*$ to ρ^* everywhere else except at $x=0$.

5.3. Proof of Corollary 2.3

Let the initial distribution of the process be the i.i.d. product measure α_0 with density $\alpha_0\{\eta_i = 1\} = \rho$, with ρ outside the disturbed range $(\rho^*, 1-\rho^*)$. Let α_t be the distribution of the process at time t. Let μ be a limit point of the time averages of α_t, so for some sequence $t_k \nearrow \infty$,

$$\mu = \lim_{k \to \infty} \frac{1}{t_k} \int_0^{t_k} \alpha_s\,ds$$

in the weak sense on the compact state space $\{0, 1\}^{\mathbf{Z}}$. Such a limit point exists by compactness, and is then automatically invariant for the process.

Write E^{α_0} for expectation under the path measure of the process started with distribution α_0. For $i \neq 0$ we have the equation

$$E^{\alpha_0}[J_i(t)] = \int_0^t \alpha_s\{\eta_i = 1, \eta_{i+1} = 0\}\, ds \tag{5.8}$$

because $J_i(t)$ [= the number of jumps from site i up to time t] increases by 1 at rate 1 when the event inside the braces holds. Thus

$$\begin{aligned} \mu\{\eta_i = 1, \eta_{i+1} = 0\} &= \lim_{k \to \infty} \frac{1}{t_k} \int_0^{t_k} \alpha_s\{\eta_i = 1, \eta_{i+1} = 0\}\, ds \\ &= \lim_{k \to \infty} \frac{1}{t_k} E^{\alpha_0}[J_i(t_k)] \\ &= \rho(1-\rho) \end{aligned} \tag{5.9}$$

The last equality is from a combination of (4.3), the limits (4.12)–(4.13), the ordering of the z_i's, and Case 1 of Corollary 2.1 with initial function $v_0(x) = \rho x$. $J_i(t)$ is bounded by a Poisson(t) random variable, hence there is uniform integrability to justify the limit of the expectation. If $i = 0$, we have to multiply the right-hand side of (5.8) by the factor r, so the limit on the last line of (5.9) is multiplied by r^{-1}. This proves (2.21) for μ.

Suppose the initial distribution α_0 has a macroscopic profile ρ_0 in the sense of (2.16) such that $\rho^* \leqslant \rho_0(x) \leqslant 1 - \rho^*$. Then the limit in (5.9) is valid with $\rho = \rho^*$. In particular, this is the case for the α_0 with non-entropy shock described in the second paragraph of Corollary 2.3.

ACKNOWLEDGMENTS

I thank Maury Bramson, Pablo Ferrari, Joel Lebowitz, Tom Liggett, Fraydoun Rezakhanlou, and Herbert Spohn for discussions on this and related problems, and Dan Ostrov for guidance on Hamilton-Jacobi equations with discontinuities. Research partially supported by NSF Grant DMS-9801085.

REFERENCES

1. S. A. Janowsky and J. L. Lebowitz, Finite size effects and shock fluctuations in the asymmetric simple exclusion process, *Phys. Rev. A* **45**:618–625 (1992).
2. S. A. Janowsky and J. L. Lebowitz, Exact results for the asymmetric simple exclusion process with a blockage, *J. Statist. Phys.* **77**:35–51 (1994).

3. T. Seppäläinen, Hydrodynamic scaling, convex duality, and asymptotic shapes of growth models, *Markov Process. Related Fields* **4**:1–26 (1998).
4. P. Covert and F. Rezakhanlou, Hydrodynamic limit for particle systems with nonconstant speed parameter, *J. Statist. Phys.* **88**:383–426 (1997).
5. T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Springer-Verlag, New York, 1999).
6. C. Landim, Hydrodynamical limit for space inhomogeneous one-dimensional totally asymmetric zero-range processes, *Ann. Probab.* **24**:599–638 (1996).
7. H. Rost, Non-equilibrium behaviour of a many particle process: Density profile and local equilibrium, *Z. Wahrsch. Verw. Gebiete* **58**:41–53 (1981).
8. C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer-Verlag, New York, 1999).
9. P. W. Glynn and W. Whitt, Departures from many queues in a series, *Ann. Appl. Probab.* **1**:546–572 (1991).
10. T. Seppäläinen, Existence of hydrodynamics for the totally asymmetric simple K-exclusion process, *Ann. Probab.* **27**:361–415 (1999).
11. J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Royal Stat. Soc. Ser. B* **30**:499–510 (1968).
12. L. C. Evans, *Partial Differential Equations* (American Mathematical Society, 1998).
13. D. Ostrov, Solutions of Hamilton-Jacobi equations and scalar conservation laws with discontinuous space-time dependence, preprint (1999).
14. T. Seppäläinen, Coupling the totally asymmetric simple exclusion process with a moving interface, *Markov Process. Related Fields* **4**:593–628 (1998).
15. K. Johansson, Shape fluctuations and random matrices, *Comm. Math. Phys.* **209**:437–476 (2000).
16. R. L. Wheeden and A. Zygmund, *Measure and Integral* (Marcel Dekker, 1977).